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Algebraic identities among the infinitesimal generators of the orthogonal and symplectic groups

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Abstract. Following Okubo's methods for the U(n) groups, we derive a hierarchy of identities satisfied by the infinitesimal generators of the orthogonal and the symplectic groups in any irreducible representation. These identities were previously derived by Green using methods developed by Bracken and Green for the solution of the problem for the unitary groups. However, we are able also to establish the *minimality* of these identities and that these in turn lead to the expression of a tensor operator in any irreducible representation of their groups as a polynomial of known degree in terms of the infinitesimal generators. The connection between the Perelomov and Popov matrices and these identities is mentioned to suggest that these matrices might play a fundamental role in the understanding of the semi-simple Lie groups.

1. Introduction

Okubo (1975) obtained a hierarchy of identities which the infinitesimal generators of the semi-simple unitary groups U(n) satisfy in any irreducible unitary representation. These identities were discovered earlier by Bracken and Green (1971) using different methods. The advantage in Okubo's method is that we are also able to prove that these identities are minimal. However, to establish this result, one requires the analytic expression for certain sums needed in the computation of the eigenvalues of the Casimir operators of these groups.

Recently, we (Nwachuku and Rashid 1976, to be referred to as I) obtained closed and analytic expressions for the corresponding sums which are required in the calculation of the eigenvalues of the Casimir operators for the orthogonal and the symplectic groups, thus unifying the technique for performing these computations for the various series of the semi-simple Lie groups. In the present paper we have followed Okubo's methods to derive the corresponding identities which the infinitesimal generators of the orthogonal and the symplectic groups satisfy in any irreducible representation. These identities were also obtained earlier by Green (1971). However, with our knowledge of the sums referred to above we are able, in addition, to show that these identities are indeed minimal.

Furthermore we can establish that any tensor operator, in any irreducible representation of these groups, can be expressed as a polynomial in the infinitesimal generators. For the degenerate irreducible representations, the degree of the polynomials in the identities as well as in the expression for a general tensor operator in terms of the generators, can be reduced. This reduction is explained in the paper.

Thus we have been able to present a unified picture of the solution to this very important problem.

Also we are able to connect the identities with the matrix of Perelomov and Popov (1966). In fact the identities contain eigenvalues of this upper triangular matrix whereas the eigenvalue C_p of the Casimir operator of pth order is just the sum of the elements of the pth power of this matrix. It seems that their matrix plays a fundamental role in the understanding of these semi-simple Lie groups.

This paper is organised in the form of theorems and proofs which are broadly divided into main sections: after the mathematical preliminaries of § 2, § 3 deals with a series of theorems leading to the required algebraic identities among the infinitesimal generators. The minimality together with the uniqueness of these identities is treated in § 4. Section 5 concludes with the polynomial expansion of tensor operators in the infinitesimal generators. We have indicated clearly where a result is valid for a general tensor operator which need not be a generator.

2. Mathematical preliminaries

We shall deal with the orthogonal groups O($2n$), O($2n+1$) and the symplectic groups Sp($2n$) together.

The infinitesimal generators $G^i{}_j$ (the indices i, j run from $-n$ to n and exclude zero for both O($2n$) and Sp($2n$)) of the orthogonal and symplectic groups satisfy the commutation relations

$$[G^i{}_j, G^k{}_l] = \delta^k_j G^i{}_l - \delta^i_l G^k{}_j + \epsilon_i\epsilon_j(\delta^l_{-j} G^k{}_{-i} - \delta^{-i}_k G^{-j}{}_l) \tag{1}$$

where the symbols ϵ_i are 1 for O(N) (N will always mean $2n$ or $2n+1$) and take the values $1, 0, -1$ for $i>0$, $i=0$, $i<0$, respectively, for the group Sp($2n$). The infinitesimal generators $G^i{}_j$ possess the symmetry

$$G^i{}_j = -\epsilon_i\epsilon_j G^{-j}{}_{-i} \tag{2}$$

which shows that there are only $\frac{1}{2}N(N-1)$ and $n(n+1)$ independent infinitesimal generators of the groups O(N) and Sp($2n$), respectively. All these groups are of rank n and the n generators $G^i{}_i$ 0) are the *diagonal* mutually commuting generators.

A set of quantities $\{T^i{}_j\}$ which satisfy the commutation relations

$$[G^i{}_j, T^k{}_l] = \delta^k_j T^i{}_l - \delta^i_l T^k{}_j + \epsilon_i\epsilon_j(\delta^l_{-j} T^k{}_{-i} - \delta^{-i}_k T^{-j}{}_l) \tag{3}$$

form a tensor operator.

Any irreducible representation of these groups is characterised by a Young tableau of, in general, n rows with $f_n, f_{n-1}, \ldots, f_1$ blocks ($f_n \geqslant f_{n-1} \ldots \geqslant f_1$) wherein we have numbered the rows as $n, n-1, \ldots, 1$ from above. Each state has a weight which consists of the eigenvalues which the diagonal generators have on that state. The highest state $|\phi\rangle$ is characterised by

$$G^i{}_i|\phi\rangle = f_i|\phi\rangle, \qquad i>0; \tag{4}$$

obviously then

$$G^{-i}{}_{-i}|\phi\rangle = -f_i|\phi\rangle = f_{-i}|\phi\rangle \tag{5}$$

where we have defined $f_{-i} = -f_i$ which is consistent with $G^0{}_0 = 0$ for the O($2n+1$) since $f_{-i} = -f_i$ implies $f_0 = 0$.

3. Derivation of identities

Theorem 1. For any tensor operator (with the ordering $n, n-1, \ldots, -n$) $T^i{}_j|\phi\rangle = 0$ for $i > j$.

Proof. Let us first examine the case $k > l \geq 0$. From the commutation relations (3) we obtain

$$[G^i{}_i, T^k{}_l] = (\delta_i^k - \delta_l^i + \delta_i^{-l} - \delta_{-i}^k)T^k{}_l,$$

which shows that $G^i{}_i$ commutes with $T^k{}_l$ for $i > k$, whereas $[G^k{}_k, T^k{}_l] = T^k{}_l$. Thus either $T^k{}_l|\phi\rangle$ has the weight $f_n, f_{n-1}, \ldots, f_{k+1}, f_k + 1, \ldots$ which is higher than the highest weight, namely $f_n, f_{n-1}, \ldots, f_1$, which is impossible, or $T^k{}_l|\phi\rangle = 0$ for $k > l \geq 0$. (The weights are ordered such that when we compare two weights, we compare their components in the order $n, n-1, \ldots$, and call a weight higher than another if the first unequal component has a higher value for that weight.)

Now when $l < 0$, if $k > l$ and $|k| > |l|$ the argument above carries over trivially, if $k > l$ and $|k| \leq |l|$ then $-l > -k$ where $-l > 0$, and the same argument implies that $T^k{}_l|\phi\rangle$ has weight $f_n, f_{n-1}, \ldots, f_{-l+1}, -f_{-l}+1$ (or $f_{-l}+2$ if $k = -l$)$\ldots$, which is also higher than the weight of the highest state. Thus, in every case $T^k{}_l|\phi\rangle = 0$ whenever $k > l$. Note that this is a property of the tensor operator T satisfying the commutation relation (3) and does not depend upon whether it is constructed from the generators or not.

Theorem 2. If for an irreducible representation $f_i = f_j$ $(i \neq j)$ and $i > j$ then

$$T^j{}_i|\phi\rangle = 0 \qquad (i, j \text{ are either both positive or both negative}).$$

Proof. $T^j{}_i|\phi\rangle$ has the weight $f_n, f_{n-1}, \ldots, f_i - 1, \ldots, f_j + 1, \ldots, f_1$ if $i > j > 0$ (or the weight $f_n, f_{n-1}, \ldots, f_{-j} - 1, \ldots, f_{-i} + 1, \ldots$ if $0 > i > j$) which is not a weight in an irreducible representation having a tableau with $f_i = f_j$ (then also $f_i = f_{i-1} = \cdots = f_j$) as can be seen immediately from the symmetry of the diagram and the characterisation of the highest state which fills the ith row by the symbol i.

Theorem 3. Defining $T^i{}_i|\phi\rangle = \sigma_i(T)|\phi\rangle$,

$$\sigma_i(G^n T)|\phi\rangle = \sum_j K_{ij}\sigma_j(G^{n-1}T)|\phi\rangle, \tag{6}$$

where the Perelomov and Popov matrix $\mathbf{K}$ is given by

$$K_{ij} = \begin{cases} [f_i + n + i - \frac{1}{2}(1+e_i)]\delta_{ij} - \theta_{ij} + \frac{1}{2}(1+e_i)\delta_{i,-j} & \text{for O}(2n+1), \\ (f_i + n + i - 1 - e_i)\delta_{ij} - \theta_{ij} + \frac{1}{2}(1+e_i)\delta_{i,-j} & \text{for O}(2n), \\ (f_i + n + i)\delta_{ij} - \theta_{ij} - \frac{1}{2}(1+e_i)\delta_{i,-j} & \text{for Sp}(2n). \end{cases} \tag{7}$$

Here $\theta_{ij} = 0, 1$ for $i \leq j$, $i > j$, respectively, $e_i = 1, 0, -1$ for $i > 0$, $i = 0$, $i < 0$ respectively, and the product of operators is defined by generalising

$$(ST)^i{}_j = \sum_k S^i{}_k T^k{}_j. \tag{8}$$

Proof.

$$(G^nT)^i{}_i|\phi\rangle = \sum_j G^i{}_j(G^{n-1}T)^j{}_i|\phi\rangle.$$

Since $G^{n-1}T$ is a tensor operator, the sum in the above equation, on account of theorem 1, is restricted to $j \leq i$. We separate the term with $j=i$ and write

$$(G^nT)^i{}_i|\phi\rangle = G^i{}_i(G^{n-1}T)^i{}_i|\phi\rangle + \sum_{j<i} G^i{}_j(G^{n-1}T)^j{}_i|\phi\rangle.$$

Adding to the jth term in the sum on the right $-(G^{n-1}T)^j{}_iG^i{}_j|\phi\rangle$ which is zero since $i>j$ we arrive at (everywhere in the following, the upper and lower signs refer to O(N) and Sp($2n$) respectively)

$$\begin{aligned}(G^nT)^i{}_i|\phi\rangle &= G^i{}_i(G^{n-1}T)^i{}_i|\phi\rangle + \sum_{j<i}[G^i{}_j, (G^{n-1}T)^j{}_i]|\phi\rangle \\ &= G^i{}_i(G^{n-1}T)^i{}_i|\phi\rangle + \sum_{j<i}(1 \mp \delta_j^{-i})[(G^{n-1}T)^i{}_i - (G^{n-1}T)^j{}_j]|\phi\rangle \\ &= \left(f_i\sigma_i(G^{n-1}T) + \sum_{j<i}[(1\mp\delta_j^{-i})(\sigma_i(G^{n-1}T) - \sigma_j(G^{n-1}T))]\right)|\phi\rangle\end{aligned}$$

or

$$\sigma_i(G^nT)|\phi\rangle = \sum_j K_{ij}\sigma_j(G^{n-1}T)|\phi\rangle$$

where the matrix **K** is as given in the theorem.

We shall designate the diagonal matrix element K_{ii} of the matrix **K** by λ_i. Then the eigenvalues are as follows:

O($2n+1$):
$$\lambda_i = \begin{cases} f_i+n+i-1, & 1 \leq i \leq n \\ n, & i=0 \\ f_i+n+i, & -n \leq i \leq -1 \end{cases}$$

O($2n$):
$$\lambda_i = \begin{cases} f_i+n+i-2, & 1 \leq i \leq n \\ f_i+n+i, & -n \leq i \leq -1 \end{cases}$$

Sp($2n$):
$$\lambda_i = f_i+n+i, \qquad -n \leq i \leq n \quad (i \neq 0).$$

This result can be used repeatedly to arrive at

$$\sigma_i(G^nT)|\phi\rangle = \sum_j (K^n)_{ij}\sigma_j(T)|\phi\rangle. \tag{9}$$

Theorem 4. If $T^i{}_j|\phi\rangle = 0$ for all $j \leq k$ and for all i, and if $f_k = f_{k+1}$, then $T^i{}_j|\phi\rangle = 0$ for all $j \leq k+1$.

Proof. We need to prove the result only where $j=k+1$. Now

$$[G^k{}_{k+1}, T^i{}_k] = \delta^i_{k+1}T^k{}_k - \delta^k_k T^i{}_{k+1} + \epsilon_k\epsilon_{k+1}(\delta^{-k}_{k+1}T^i{}_{-k} - \delta^i_{-k}T^{-(k+1)}{}_k).$$

Operating on $|\phi\rangle$ we see that the left-hand side is zero since $G^k{}_{k+1}|\phi\rangle = 0$ (theorem 2 with $f_k = f_{k+1}$) and $T^i{}_k|\phi\rangle = 0$ on account of the condition given in the theorem. The same conditions in the theorem also make the first and the last terms on the right-hand side vanish. The term containing δ^{-k}_{k+1} is zero since $-k = k+1$ requires $k = -\frac{1}{2}$. Thus we are left with $T^i{}_{k+1}|\phi\rangle = 0$ for all i, which is what we required to prove.

Theorem 5.

$$(G^n T)^i{}_j = (TG^n)^i{}_j \qquad \text{and} \qquad (G^n \circ T)^i{}_j = (T \circ G^n)^i{}_j$$

where the $\circ$ product is defined by generalising

$$(S \circ T)^i{}_j = \sum_k S^k{}_j T^i{}_k.$$

Proof. We shall prove $(GT)^i{}_j = (TG)^i{}_j$, from which the first result immediately follows. The second result is proved analogously.

Since $\Sigma_i (G^2)^i{}_i = \Sigma_{i,j} G^i{}_j G^j{}_i$ is a Casimir operator, it commutes with every tensor operator. Now computing $[\Sigma_i (G^2)^i{}_i, T^j{}_k]$ and using $G^i{}_j = -\epsilon_i \epsilon_j G^{-j}{}_{-i}$ we obtain

$$\left[\sum_i (G^2)^i{}_i, T^j{}_k\right] = 2[(T \circ G)^j{}_k - (TG)^j{}_k + (GT)^j{}_k - (G \circ T)^j{}_k] = 0.$$

Also we can immediately show that

$$[G^j{}_i, T^i{}_k] + [G^i{}_k, T^j{}_i] = (GT)^j{}_k - (T \circ G)^j{}_k + (G \circ T)^j{}_k - (TG)^j{}_k = 0$$

or

$$\left[\sum_i (G^2)^i{}_i, T^j{}_k\right] = 4[(GT)^j{}_k - (TG)^j{}_k] = 0$$

or

$$(GT)^j{}_k = (TG)^j{}_k.$$

We note that the above theorem on the commutativity of the generators and any tensor operator in products is the result of the commutation relations only and not of operation on the highest state.

Theorem 6. If $T^i{}_j|\phi\rangle = 0$ for all i, j, then $T^i{}_j = 0$.

Proof. $T^i{}_j = 0$ means that this tensor operator annihilates every state of an irreducible representation. To prove the result we just note that the states of the representation can be obtained from the highest state by the repeated application of the generators. On commuting $T^i{}_j$ through them, we have $T^i{}_j$ operating on the highest state which it is given to annihilate. Hence it will annihilate every state or $T^i{}_j = 0$.

Theorem 7. If $T^i{}_j|\phi\rangle = 0$ for all $j \leq k$ and all i, then $\Sigma_l (G - \lambda_{k+1} I)^i{}_l T^l{}_j|\phi\rangle = 0$ for all $j \leq k+1$, and for all i.

Proof. On account of the commutation property mentioned in theorem 5

$$\sum_l (G - \lambda_{k+1} I)^i{}_l T^l{}_j|\phi\rangle = \sum_l T^i{}_l (G - \lambda_{k+1} I)^l{}_j|\phi\rangle.$$

We must prove that the above expressions are zero for all i and for all $j \leqslant k+1$ where we know that $T^i{}_j|\phi\rangle = 0$ for all i and for all $j \leqslant k$.

To prove our result we note that if $j \leqslant k$, we can use the expression on the left to establish our result. Thus we need to prove the result when $j = k+1$. For this purpose we utilise the expression on the right when $j = k+1$. We separate the term with $l = j = k+1$, and in the remaining terms $\lambda_i I$ does not contribute. Thus we see that

$$\sum_l (G - \lambda_{k+1} I)^i{}_l T^l{}_{k+1}|\phi\rangle$$

$$= \sum_l T^i{}_l (G - \lambda_{k+1} I)^l{}_{k+1}|\phi\rangle$$

$$= T^i{}_{k+1}(G - \lambda_{k+1} I)^{k+1}{}_{k+1}|\phi\rangle + \sum_{l<k+1} T^i{}_l G^l{}_{k+1}|\phi\rangle.$$

Add to the lth term on the right-hand side $-G^l{}_{k+1} T^i{}_l|\phi\rangle$ which is zero since $l < k+1$. Thus we have

$$\sum_l (G - \lambda_{k+1} I)^i{}_l T^l{}_{k+1}|\phi\rangle$$

$$= (f_{k+1} - \lambda_{k+1}) T^i{}_{k+1}|\phi\rangle - \sum_{l<k+1} [G^l{}_{k+1}, T^i{}_l]|\phi\rangle$$

$$= (f_{k+1} - \lambda_{k+1}) T^i{}_{k+1}|\phi\rangle$$

$$- \sum_{l<k+1} [\delta^i_{k+1} T^l{}_l - \delta^l_l T^i{}_{k+1} + \epsilon_l \epsilon_{k+1}(\delta^l_{-(k+1)} T^i{}_{-l} - \delta^{-l}_i T^{-(k+1)}{}_l)]|\phi\rangle.$$

From the summation on the right, the first and last terms vanish since $l < k+1$ implies $l \leqslant k$. The above expression now simplifies to

$$\left((f_{k+1} - \lambda_{k+1}) + \sum_{l<k+1} (1 \mp \delta^l_{-(k+1)})\right) T^i{}_{k+1}|\phi\rangle$$

which will vanish since $[f_{k+1} + \Sigma_{l<k+1} (1 \mp \delta^l_{-(k+1)})]$ is exactly the $K_{k+1,k+1} = \lambda_{k+1}$ diagonal matrix element of the matrix $\mathbf{K}$ in (7).

Theorem 8. The infinitesimal generators $G^i{}_j$ satisfy the identity

$$\left[\prod_{k=-n}^{n} (G - \lambda_k I)\right]^i{}_j = 0 \qquad (10)$$

for the irreducible representation given by $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ where if $f_l = f_{l+1}$, the factor $(G - \lambda_{l+1} I)$ can be omitted.

Proof. Note that on account of the commutation property (theorem 5) the factors in the above identity may be placed in any order. Also using theorem 6, we only need to prove that the operator product in the statement of the theorem annihilates the highest state $|\phi\rangle$ in order to prove that this operator is identically zero. To prove this result we write it in the order $[(G - \lambda_n I)(G - \lambda_{n-1} I) \ldots (G - \lambda_{-n} I)]^i{}_j$ and prove that it annihilates the highest state. Now $\lambda_{-n} = f_{-n}$ for each of the groups (see equation (7)). We show that $(G - \lambda_{-n} I)^i{}_j|\phi\rangle = 0$ for all i and $j \leqslant -n$, which implies that $j = -n$. Now using theorem 1 we need to observe the result only for $i \leqslant -j = -n$, i.e. for $i = -n$. Finally

$$(G - \lambda_{-n} I)^{-n}{}_{-n} |\phi\rangle = (f_{-n} - \lambda_{-n}) |\phi\rangle = 0.$$

Now we can use induction, which holds on account of theorem 7 and arrive at the identities. Using theorem 4, we also establish the omission of the factor $(G - \lambda_{l+1} I)$ whenever $f_l = f_{l+1}$.

In the above theorem we have obtained the identities we were interested in. We note that the identities in equation (10) contain the eigenvalues of the Perelomov and Popov (1966) matrix **K** given above. In our previous paper I on the computation of the eigenvalues of the Casimir operators of these groups, we noticed that the eigenvalue C_p of the Casimir operator of order p is just the sum of the pth power of the elements of this matrix. Thus this matrix seems to be playing a fundamental role in the understanding of these groups.

Theorem 9.

$$\left[\prod_{k=-n}^{n} \circ (G + \lambda_k I)\right]^i_j = 0$$

where, if $f_l = f_{l+1}$, we can omit the factor $(G + \lambda_{l+1} I)$ from the product. The $\circ$ in front of the product indicates that we are using the other type of product, namely

$$(TS)^i{}_j = \sum_k T^k{}_j S^i{}_k$$

Proof. Noting that

$$(G + \lambda I)^i{}_j = -\epsilon_i \epsilon_j (G - \lambda I)^{-j}{}_{-i},$$

we observe that

$$\left[\prod_{k=-n}^{n} (G - \lambda_k I)\right]^i_j = 0 \quad \Rightarrow \quad (-)^p \epsilon_i \epsilon_j \left[\prod_{k=-n}^{n} \circ (G + \lambda_k I)\right]^i_j = 0,$$

where p is the number of factors in the product, or

$$\left[\prod_{k=-n}^{n} \circ (G + \lambda_k I)\right]^i_j = 0$$

where the rules of omission of factors are precisely those for the other type of product.

This theorem could also be proved by repeating the arguments in the previous theorems and making appropriate changes.

4. Minimality and uniqueness

Theorem 10. The identities contained in theorem 8 are of minimal degree and unique up to multiplication by a non-zero number.

Proof. The uniqueness follows immediately from minimality. To establish minimality we attempt to diagonalise the matrix **K**. To prove our result, in fact, we only need the sum of the components of the left eigenvectors when we take the first non-zero element as unity. The diagonalisation of the **K** matrix has been carried out elsewhere and the relevant sums are computed therein (see I). Using the notation $\mathbf{X}^{-1}\mathbf{K}\mathbf{X} = \mathbf{K}_D$ or $\mathbf{K} = \mathbf{X}\mathbf{K}_D\mathbf{X}^{-1}$, where $\mathbf{K}_D = \delta_{ij}K_{ii} = \delta_{ij}\lambda_i$, we deduce from theorem 3 that $\sigma_i(G^nT) = \Sigma_j (K^n)_{ij}\sigma_j(T) = \Sigma_{j,l} (X)_{il}\lambda^n{}_l(X^{-1})_{lj}\sigma_j(T)$ for any tensor operator T. Or, for any polynomial function $f(G)$ of the generators,

$$\sigma_i(f(G)) = \sum_{j,l} X_{il}f(\lambda_l)(X^{-1})_{lj}\sigma_j(T). \tag{11}$$

If $f(G) = 0$ identically, putting $T = I$ in the above equation, we require

$$\sum_{i \geq l \geq j} X_{il}f(\lambda_l)(X^{-1})_{lj} = 0 \qquad \text{for all } i, \tag{12}$$

where the range on the summation is dictated on account of the diagonalising matrix **X** as well as its inverse being upper triangular as **K** itself is.

The sums $\Sigma_j (X^{-1})_{ij}$ have been computed in I and are given by:

$$-\sum_j (X^{-1})_{ij} = \begin{cases} \left\{\begin{matrix} \dfrac{(1+\lambda_0-\lambda_i)(1+\lambda_{-i}-\lambda_i)-(\lambda_0-\lambda_i)}{\lambda_{-i}-\lambda_i} \\ 2+\lambda_{-i}-\lambda_i \end{matrix}\right\} \dfrac{\Pi'^{i-1}_{j=-n}(1+\lambda_j-\lambda_i)}{\Pi^{i-1}_{j=-n}(\lambda_j-\lambda_i)}, & i \geq 0 \quad (13) \\ \dfrac{\Pi^{i-1}_{j=-n}(1+\lambda_j-\lambda_i)}{\Pi^{i-1}_{j=-n}(\lambda_j-\lambda_i)}, & i \leq 0 \quad (14) \end{cases}$$

where the prime on the product in (13) indicates that $j = 0$ and $j = -1$ are to be omitted, and the factor on the outside has been given for the three cases $O(2n+1)$, $O(2n)$, $Sp(2n)$. Note that the only factor which can vanish in the above sums is $(1+\lambda_{i-1}-\lambda_i)$, which vanishes when $\lambda_i = 1+\lambda_{i-1}$ or when $f_i = f_{i-1}$ except when $i = 1$. Since (12) holds for all i, we start with $i = -n$ when $l = -n$, $X_{-n,-n}(=X_{ii}) = 1$ (see I). This is in fact the choice of the leading non-zero component and $\Sigma_j (X^{-1})_{-nj} = 1$. Thus $f(\lambda_{-n}) = 0$. We now consider $i = -n+1$ and use $f(\lambda_{-n}) = 0$, which gives $l = -n+1$ and we notice that $f(\lambda_{-n+1}) = 0$ except when $\lambda_{-n+1} = 1+\lambda_{-n}$ which is equivalent to $f_{-n+1} = f_{-n}$. Thus $f(G)$ will have λ_{-n} as a factor and also λ_{-n+1} if $f_{-n+1} \neq f_{-n}$. This process can be continued to prove the minimality of our identities.

Remarks.

(1) In view of the minimality and uniqueness established in this theorem, the identities in theorem 9 must be the same as those previously obtained in theorem 8.

(2) Theorems 7 and 10 show that, in general, $(G^{2n})^i_j$, (or $(G^{2n+1})^i_j$ for $O(2n+1)$) can be expressed in terms of lower powers of the generators in a unique manner in any irreducible representation of these groups. If however, the representation has some rows equal, a known lower power of G can be expanded in terms of the lower powers.

5. Expansion in the infinitesimal generators

Now we come to the expansion of any tensor operator as a polynomial in the infinitesimal generators.

Theorem 11. For any tensor operator $T^i{}_j$, the following three statements are equivalent:

(i) $T^i{}_j = 0$ identically;

(ii) $\sigma_i(T) = 0$ for all $i = -n, \ldots, n$;

(iii) $\langle G^j T\rangle = \sum_{i=-n}^{n} \sigma_i(G^j T) = 0$ for $j = 0, 1, 2, \ldots$.

Proof. The statement (ii) immediately follows from (i). Again (iii) follows from (ii) using equation (9). We now show that (ii) follows from (iii). If $f(G)$ is *any* polynomial in the generators, equation (11) gives, using (iii),

$$\sum_i \sigma_i(f(G)T) = \sum_{i,l,j} X_{il} f(\lambda_l)(X^{-1})_{lj}\sigma_j(T) = 0.$$

Here the sum $\Sigma_i X_{il}$ of the elements in the lth column of the diagonalising matrix is given in I. This is indeed

$$\sum_i X_{il} = \begin{cases} \dfrac{\Pi_{j=i+1}^{n}(1+\lambda_j-\lambda_i)}{\Pi_{j=i+1}^{n}(\lambda_j-\lambda_i)}, & i \geqslant 0, \quad (15) \\[2ex] \left\{\begin{matrix} (1+\lambda_0-\lambda_i)(1+\lambda_{-i}-\lambda_i)-(\lambda_0-\lambda_i) \\ \lambda_{-i}-\lambda_i \\ 2+\lambda_{-i}-\lambda_i \end{matrix}\right\} \dfrac{\Pi'^{n}_{j=i+1}(1+\lambda_j-\lambda_i)}{\Pi_{j=i+1}^{n}(\lambda_j-\lambda_i)}, & i \leqslant 0 \quad (16) \end{cases}$$

where the prime on the product in (16) indicates that $j = 0$ and $-i$ are to be omitted and the outside bracket is given for $O(2n+1)$, $O(2n)$, $Sp(2n)$, respectively. These products are in general non-vanishing. Thus we can follow Okubo and deduce our results by taking in turn special functions f such that $f(l_i) = 0$ for all i except $i = j$.

Finally we must obtain (i) from (ii), i.e. from $\sigma_i(T) = 0 = T^i{}_i|\phi\rangle$ for all i, we wish to prove that $T^i{}_j = 0$ identically, for which (see theorem 4) we need to prove only $T^i{}_j|\phi\rangle = 0$ for all i and j. Since, in particular $T^{-n}{}_{-n}|\phi\rangle = 0$ using theorem 7 repeatedly, we arrive at the identity

$$[(G-\lambda_n I)(G-\lambda_{n-1}I)\ldots(G-\lambda_{-n+1}I)T]^i{}_j|\phi\rangle = 0 \tag{17}$$

for all i, j. Note that this identity differs from the identity for G in the replacement of the factor $(G-\lambda_{-n}I)$ by T. We now use the fact that the commutation relation (3) are invariant when we write

$$\tilde{T}^i{}_j = -\epsilon_i\epsilon_j T^{-j}{}_{-i} \quad \text{and} \quad \tilde{G}^i{}_j = -\epsilon_i\epsilon_j G^{-j}{}_{-i},$$

which also requires

$$\tilde{f}_i|\phi\rangle = \tilde{G}^i{}_i|\phi\rangle = -G^{-i}{}_{-i}|\phi\rangle = f_i,$$

i.e. the characterisation of the irreducible representation does not alter by this transformation. This transformation on (17) gives us

$$[(G+\lambda_n I)\circ(G+\lambda_{n-1}I)\circ\ldots\circ(G+\lambda_{-n+1}I)\circ T]^i{}_j|\phi\rangle = 0 \tag{18}$$

for all i, j where we now have the other type of product. In order to compare (18) with (17), we try to recast (18) in the form of (17). For this purpose note that

$$[(G+aI)\circ S]^k{}_l|\phi\rangle = 0,$$

where $k \geqslant l$, and S is some polynomial in G multiplied by T. Also

$$[(G+aI)\circ S]^k{}_l|\phi\rangle = \sum_m (G+aI)^m{}_l S^k{}_m|\phi\rangle.$$

But

$$\begin{aligned}[(G+aI)^m{}_l, S^k{}_m]|\phi\rangle \\ &= [\delta^k_l S^m{}_m - \delta^m_m S^k{}_l + \epsilon_m\epsilon_l(\delta^m_{-l}S^k{}_{-m} - \delta^k_{-m}S^{-l}{}_m)]|\phi\rangle \\ &= (\delta^k_l S^m{}_m - \delta^m_m S^k{}_l + \epsilon_l\epsilon_{-l}\delta^m_{-l}S^k{}_l - \epsilon_{-k}\epsilon_l\delta^k_{-m}S^{-l}{}_{-k})|\phi\rangle.\end{aligned}$$

Here $S^m{}_m|\phi\rangle = 0$ and since $k < l$, $-l > -k$, we also have $S^{-l}{}_{-k}|\phi\rangle = 0$. Thus we arrive at

$$\begin{aligned}[(G+aI)\circ S]^k{}_l|\phi\rangle \\ &= [S(G+aI)]^k{}_l|\phi\rangle - \sum_m (1 \mp \delta^m_{-l})S^k{}_l|\phi\rangle \\ &= \left[S\left(G+aI-\begin{Bmatrix}2n\\2n-1\\2n+1\end{Bmatrix}I\right)\right]^k_l|\phi\rangle\end{aligned}$$

where the direction of the product has now been reversed. Using this equation repeatedly on (18) we obtain for each group

$$[(G-(\lambda_{n-1}+1)I)(G-(\lambda_{n-2}+1)I)\dots(G-(\lambda_{-n}+1)I)T]^i{}_j|\phi\rangle = 0.$$

We note that here the factor corresponding to λ_n is missing. We remark also that we cannot use these methods to prove that the two identities for the generators may be different. In fact (18) has been obtained using $\sigma_i(T) = 0$ for all i, which for the generators means that we are considering the trivial representation.

The identities (17) and (19) are not the same. In fact none of the factors in (17) coincides with any factor in (19). Note that we are considering the case where the f_i are all different. If any two of them had been equal, we would have omitted the corresponding factors from the two identities.

These identities finally give $T^i{}_j|\phi\rangle = 0$ for all i, j or $T^i{}_j = 0$.

The equivalence of (i) and (iii) in theorem 11 implies that any tensor $T^i{}_j$ in an irreducible representation can be expanded as a polynomial (of degree equal to one less than the order of the group) of the infinitesimal generators of the group. The degree of the polynomial will be reduced if some of the rows of the Young tableau for the representation are equal.

References

Bracken A J and Green H S 1971 *J. Math. Phys.* **12** 2099–106
Green H S 1971 *J. Math. Phys.* **12** 2107–13
Nwachuku C O and Rashid M A 1976 *J. Math. Phys.* **17** 1611–6
Okubo S 1975 *J. Math. Phys.* **16** 528–35
Perelomov A M and Popov V S 1966 *J. Nucl. Phys., USSR* **3** 1127–34